17

COLLIDING ELECTROMAGNETIC WAVES: EXACT SOLUTIONS

It is the purpose of this chapter and the following one to review the exact solutions which describe colliding plane electromagnetic waves or combinations of gravitational and electromagnetic waves that are presently known. Since these space-times have two Killing vectors and an Ernst method can be used, there are a whole range of generating techniques that can be applied to obtain new classes of solutions. It is therefore not possible to review exact solutions without also reviewing the generating techniques that have been applied to obtain them. The techniques described may clearly be used to obtain further solutions. This chapter reviews classes of non-diagonal solutions. The particular case of diagonal solutions is considered in Chapter 18.

17.1 A technique of Chandrasekhar and Xanthopoulos

Having developed the approach described in the previous chapter, Chandrasekhar and Xanthopoulos (1985*a*) then used it to obtain an electromagnetic generalization of the Nutku–Halil solution.

If the Khan–Penrose and Nutku–Halil solutions are regarded as the analogues of the Schwarzschild and Kerr solutions respectively, then it is reasonable to suppose that the Kerr–Newman solution may also have a colliding wave analogue. Accordingly, it is appropriate[1] to consider the special class of solutions for which

$$H = Q(Z+1) \tag{17.1}$$

where Q is a constant that can always be chosen to be real. In this case, (16.20) and (16.21) both reduce to the same equation:

$$\begin{aligned}\tfrac{1}{2}\left((1-2Q^2)(Z+\bar{Z})-2Q^2(Z\bar{Z}+1)\right)\Big(\big((1-t^2)Z_t\big)_{,t}-\big((1-z^2)Z_z\big)_{,z}\Big)\\ =\left(1-2Q^2(\bar{Z}+1)\right)\left((1-t^2)Z_t^2-(1-z^2)Z_z^2\right).\end{aligned} \tag{17.2}$$

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[1] See Chandrasekhar (1983) section 110.

It is also convenient to consider the restriction (17.1) in terms of the associated Ernst potentials E and η given by (16.23,24). It can immediately be seen that, in this case,

$$\eta = 2Q \tag{17.3}$$

so that equation (16.25) is automatically satisfied, and (16.25) becomes

$$\begin{aligned}(1-4Q^2-E\bar{E})\Big(\big((1-t^2)E_t\big)_{,t}-\big((1-z^2)E_z\big)_{,z}\Big)\\ =-2\bar{E}\left((1-t^2)E_t^2-(1-z^2)E_z^2\right)\end{aligned} \tag{17.4}$$

which is virtually identical to (11.19).

In this case, however, it is important to notice that the functions Z and E do not contain the metric functions as in Chapter 11. Rather, through (16.19), they contain the potentials described in the previous chapter. These are now given in terms of E by

$$\Psi=\frac{(1-4Q^2-E\bar{E})}{(1-E)(1-\bar{E})},\qquad \Phi=-i\frac{(E-\bar{E})}{(1-E)(1-\bar{E})},\qquad H=\frac{2Q}{1-E}. \tag{17.5}$$

Chandrasekhar and Xanthopoulos (1985*a*) have made use of the important result that, if E_o is a solution of the vacuum Ernst equation (11.14), then

$$E = aE_o \tag{17.6}$$

is a solution of the equation (17.4) for an Einstein–Maxwell field, where

$$a=\sqrt{1-4Q^2}. \tag{17.7}$$

In view of the possible transformation (12.9), there is no loss of generality in assuming that a is real and that $0\le Q\le \frac{1}{2}$. This technique can therefore be characterized by the Ernst potentials

$$E=aE_o,\qquad \eta=\sqrt{1-a^2}. \tag{17.8}$$

This result can also be stated in the equivalent form that, if Z_o is a solution of the vacuum Ernst equation (11.8), then a new solution of the Ernst equations (16.20) and (16.21) for an Einstein–Maxwell field is given by

$$Z=\frac{(1-a)+(1+a)Z_o}{(1+a)+(1-a)Z_o},\qquad H=\frac{\sqrt{1-a^2}\,(Z_o+1)}{(1+a)+(1-a)Z_o}. \tag{17.9}$$

It follows immediately from (17.9) that the potentials for the new solution are given in terms of the vacuum potentials $Z_\mathrm{o} = \Psi_\mathrm{o} + i\Phi_\mathrm{o}$ by

$$\begin{aligned} \Psi &= \frac{2a^2}{\Omega^2}(Z_\mathrm{o} + \bar{Z}_\mathrm{o}) &&= \frac{4a^2}{\Omega^2}\Psi_\mathrm{o} \\ \Phi &= -\frac{2ia}{\Omega^2}(Z_\mathrm{o} - \bar{Z}_\mathrm{o}) &&= \frac{4a}{\Omega^2}\Phi_\mathrm{o} \end{aligned} \tag{17.10}$$

where

$$\begin{aligned} \Omega^2 &= |(1+a) + (1-a)Z_\mathrm{o}|^2 \\ &= (1+a)^2 + 2(1-a^2)\Psi_\mathrm{o} + (1-a)^2(\Psi_\mathrm{o}^2 + \Phi_\mathrm{o}^2). \end{aligned} \tag{17.11}$$

Clearly, it is possible to use this method to obtain new solutions of the Einstein–Maxwell equations starting with any vacuum solution describing colliding gravitational waves. The new solutions contain an arbitrary parameter a. Moreover, the new solutions must reduce to the original vacuum solutions in the limit as $a \to 1$.

By substituting (17.8) into (16.28), it may be observed that the boundary conditions for colliding gravitational and electromagnetic waves of this type are automatically satisfied if they were satisfied for the original vacuum solution given by E_o or Z_o, and with the same values for k_1 and k_2. This implies that the expressions for $f(u)$ and $g(v)$ are unchanged, and thus

$$U = U_\mathrm{o}. \tag{17.12}$$

Using the approach described above with (17.1) and (17.6), the metric function χ can immediately be obtained from (16.14). It is then necessary to integrate the equations (16.15) to obtain ω. Finally it is necessary to integrate (7.9) to determine the remaining metric function M.

It may be noticed, however, that the transformation (17.9) is contained in the 'Ehlers–Harrison' transformation for a particular choice of parameters (Harrison, 1968). In this case, it is possible to find a general transformation for M, so that an integration of (7.9) is not essential. But there does not appear to be any way of avoiding the integration of (16.15) to obtain ω.

17.2 Two particular examples

In this section two results are presented that have been obtained using the above technique. The first of these is a generalization of the Nutku–Halil solution and the second a generalization of the vacuum Chandrasekhar–Xanthopoulos solution.

Chandrasekhar and Xanthopoulos (1985*a*) have obtained a new colliding wave solution using the above method and taking the initial vacuum solution to be the Nutku–Halil solution described in Section 13.1. In this case, we have $Z_\mathrm{o} = \Psi_\mathrm{o} + i\Phi_\mathrm{o}$ where

$$\begin{aligned}\Psi_\mathrm{o} &= \sqrt{1-t^2}\sqrt{1-z^2}\left(\frac{(1-pt)^2+q^2z^2}{1-p^2t^2-q^2z^2}\right)\\ \Phi_\mathrm{o} &= \frac{2q}{p}\frac{(1-z^2)(1-pt)}{(1-p^2t^2-q^2z^2)}.\end{aligned}\tag{17.13}$$

These expressions may immediately be deduced from equations (13.11) and (13.17) after noting that the vacuum Chandrasekhar–Xanthopoulos and the Nutku–Halil solutions are related simply by interchanging the expressions for Ψ and Φ with those for χ and ω. In this case, it may also be noticed that a constant term in Φ_o has been altered.

The metric functions for this solution can now be obtained from (16.14) and by integrating equations (16.15) and (7.9). The integration process is described in great detail by Chandrasekhar and Xanthopoulos (1985*a*). The resulting expressions, which are the metric functions of the line element (11.4), may be written in the form

$$\begin{aligned}e^{-U} &= \sqrt{1-t^2}\sqrt{1-z^2} = 1-u^2-v^2\\ \chi &= \frac{\Omega^2}{4a^2}\frac{(1-p^2t^2-q^2z^2)}{((1-pt)^2+q^2z^2)}\\ \omega &= \frac{qz\left(4ap^2+(1-a)^2\left[1+q^2z^2+(3-z^2)(1-pt)^2\right]\right)}{2a^2p^2\left((1-pt)^2+q^2z^2\right)}\\ e^{-M} &= \frac{\Omega^2}{4a^2}\frac{(1-p^2t^2-q^2z^2)}{(1-t^2)^{1/4}(1-z^2)^{1/4}}\end{aligned}\tag{17.14}$$

where Ω is given in terms of the expressions (17.13) by (17.11).

The structure of this solution has been described in detail by Chandrasekhar and Xanthopoulos (1985*a*), with particular emphasis on the colinear case when $q=0$ and $p=1$. As usual, there is a strong curvature singularity in region IV on the hypersurface $u^2+v^2=1$, and there are fold singularities in regions II and III when $u=1$ and $v=1$. The approaching waves are a combination of step gravitational and electromagnetic components, and the wave fronts also include impulsive gravitational waves.

A second example of the application of the technique described in Section 17.1 is contained in part I of another substantial paper of Chandrasekhar and Xanthopoulos (1987*b*). This is obtained by taking the initial solution to be the vacuum Chandrasekhar–Xanthopoulos (1986*b*) solution described in Section 13.3.

In this second example, the expression for Z_o is taken to be (13.10) so that

$$\Psi_o = \frac{1 - p^2t^2 - q^2z^2}{(1-pt)^2 + q^2z^2}, \qquad \Phi_o = \frac{2qz}{(1-pt)^2 + q^2z^2}. \tag{17.15}$$

Equations (16.15) may now be solved as in the previous example, and the resulting solution may be expressed in terms of the metric functions of the line element (11.4) as

$$\begin{aligned} e^{-U} &= \sqrt{1-t^2}\sqrt{1-z^2} = 1 - u^2 - v^2 \\ \chi &= \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{a^2}\left(\frac{(1-apt)^2 + a^2q^2z^2}{1-p^2t^2-q^2z^2}\right) \\ \omega &= -\frac{q(1-z^2)}{a^2p}\left(\frac{1+a^2-2apt}{1-p^2t^2-q^2z^2}\right) + \text{const} \\ e^{-M} &= \frac{(1-apt)^2 + a^2q^2z^2}{a^2\sqrt{1-u^2}\sqrt{1-v^2}}. \end{aligned} \tag{17.16}$$

Chandrasekhar and Xanthopoulos (1987*b*) have analysed this solution and have shown that it is of algebraic type D. They have further shown that it is in fact the Kerr–Newman space-time interior to its ergo-sphere. In this sense, it can be considered to be the natural generalization of the vacuum Chandrasekhar–Xanthopoulos (1986*b*) solution, which is a Kerr space-time interior to the ergo-sphere.

The singularity structure of this solution is the same as that of the vacuum Chandrasekhar–Xanthopoulos solution as described in Section 13.3. The space-like surface on which $u^2 + v^2 = 1$ is an unstable quasiregular singularity rather than a curvature singularity.

In both of the solutions presented in this section there are impulsive gravitational wave components along the wavefronts $u = 0$ and $v = 0$. Hence these solutions do not satisfy the conditions for Tipler's theorem 8.1.

The global structure of this solution has been further considered by Hayward (1989*a*) as an extension of his analysis of the degenerate Ferrari–Ibañez solution.

17.3 Another type D solution

It has already been pointed out in Section 16.3 that the Bell–Szekeres solution can be obtained from the vacuum Chandrasekhar–Xanthopoulos solution using the simple generation technique described in Section 16.2. Chandrasekhar and Xanthopoulos (1987*a*) have subsequently obtained a generalization of the Bell–Szekeres solution by applying an Ehlers transformation to the Ernst potential that is used to generate the Bell–Szekeres solution. The resulting space-time describing the interaction region of the new solution is of algebraic type D.

It may first be recalled that, if $Z_{\rm oo}$ is a solution of the vacuum Ernst equation (11.15), then another solution is given by

$$Z_{\rm o} = \frac{Z_{\rm oo}}{1+2i\beta Z_{\rm oo}} \tag{17.17}$$

where the parameter of this Ehlers transformation has been altered from that of (12.13) by putting $c=-2\beta$. In terms of the associated Ernst function, the transformation (12.14) may be restated in the form that, if $E_{\rm oo}$ is a solution of equation (11.14), then a new vacuum solution can be obtained using the new potential

$$E_{\rm o} = \frac{E_{\rm oo} - i\beta(1+E_{\rm oo})}{1+i\beta(1+E_{\rm oo})}. \tag{17.18}$$

Now, using the simple generation technique described in Section 16.2, an electromagnetic solution of the equations (16.20) and (16.21) is given by

$$Z=1 \qquad \text{and} \qquad H=E_{\rm o} \tag{17.19}$$

where $E_{\rm o}$ is a solution of the vacuum Ernst equation (11.14). The generalized non-diagonal Bell–Szekeres solution is expressed in this way in Section 16.3 with $E_{\rm o}=pt+iqz$. A further generalization of the Bell–Szekeres solution may now be obtained by applying the Ehlers transformation (17.18) to this expression for $E_{\rm o}$. Accordingly, we now consider an electromagnetic solution given by the potentials

$$Z=1, \qquad H=\frac{E_{\rm oo} - i\beta(1+E_{\rm oo})}{1+i\beta(1+E_{\rm oo})} \tag{17.20}$$

where

$$E_{\rm oo}=pt+iqz, \qquad \text{and} \qquad p^2+q^2=1. \tag{17.21}$$

From these expressions it follows that

$$\Psi = \frac{1-p^2t^2-q^2z^2}{(1-\beta qz)^2+\beta^2(1+pt)^2}, \qquad \Phi=0. \tag{17.22}$$

The expression for χ is now determined by (16.14), and the remaining metric functions for this solution can be obtained by integrating equations (16.15) and (7.9). The details of the integration process are described by Chandrasekhar and Xanthopoulos (1987*a*), and the resulting expressions may be written in the form

$$\begin{aligned}
e^{-U} &= \sqrt{1-t^2}\sqrt{1-z^2} = 1-u^2-v^2 \\
\chi &= \frac{\sqrt{1-t^2}\sqrt{1-z^2}\,\big((1-\beta qz)^2+\beta^2(1+pt)^2\big)}{(1-p^2t^2-q^2z^2)} \\
\omega &= \frac{p(1-t^2)\big(1-2\beta qz+2\beta^2(1+p)\big)+2\beta^2q^2(1-t)(1-z^2)}{q(1-p^2t^2-q^2z^2)} \\
e^{-M} &= (1-\beta qz)^2+\beta^2(1+pt)^2
\end{aligned} \tag{17.23}$$

where the arbitrary constant in the integral for ω has been chosen such that $\omega=0$ when $t=1$.

It may be noticed that this solution reduces to the conformally flat Bell–Szekeres solution in the limit as $\beta\to 0$. It may also be observed that it does not contain the special case of a diagonal solution in which $\omega=0$, except in this Bell–Szekeres limit.

The properties of this solution have been described in detail by Chandrasekhar and Xanthopoulos (1987*a*). They have evaluated the components of the curvature tensor and shown that the space-time in the interaction region is of algebraic type D, with the repeated principal null directions of the Weyl tensor being aligned with the principal null directions of the electromagnetic field. The approaching waves in regions II and III are a combination of gravitational and electromagnetic waves, and the usual fold singularities in these regions still occur.

One particularly significant feature of this solution is that the surface in region IV on which $t=1$ does not correspond to a curvature singularity. This surface, on which the two waves mutually focus each other, is a quasiregular singularity on which the components of the curvature tensor remain bounded. In this case, there are impulsive gravitational wave components along the wavefronts $u=0$ and $v=0$, and hence the solution does not satisfy the conditions for Tipler's theorem 8.1. In this feature it is similar to the Bell-Szekeres solution, the degenerate Ferrari–Ibañez solution of Section 10.5 and the vacuum Chandrasekhar–Xanthopoulos solution described in Section 13.3.

The solution given in this section has also been independently obtained and described by Halilsoy (1988*a*). Halilsoy has also used the simplification (17.19) as described in Section 16.2, but with the initial solution $E_{\rm o}$ being that obtained from a colinear solution using the transformation (12.44) and (12.45) with $a=1$. However, as is clear from

(12.46), this can also be considered as an Ehlers transformation (12.13) or (17.17) following a rotation of the type (12.10). It can be seen that this is precisely the sequence of transformations used by Chandrasekhar and Xanthopoulos (1987*a*) as described above.

17.4 A technique of Halilsoy

A simple technique was described in Section 16.2 in which a solution $E_{\rm o}$ of the vacuum Ernst equation (11.14) was used to generate a colliding electromagnetic wave solution of the equations (16.21) and (16.20) given by $Z=1$, $H=E_{\rm o}$. This technique was included in that of the previous section. In terms of the alternative Ernst potentials, the new solution satisfying (16.25) and (16.26) is given by $E=0$, $\eta=E_{\rm o}$.

As pointed out by Halilsoy (1989*a*,*b*), this generation technique can easily be generalized. It can immediately be seen that a new colliding electromagnetic wave solution of the equations (16.25) and (16.26) can be given in terms of the alternative Ernst potentials by

$$E=aE_{\rm o}, \qquad \eta=\sqrt{1-a^2}\,E_{\rm o} \tag{17.24}$$

where a is an arbitrary parameter.

It is also sometimes convenient to express this transformation in terms of an alternative parameter α such that

$$a=\cos 2\alpha. \tag{17.25}$$

The new solution may then be expressed in the form

$$E=\cos 2\alpha\,E_{\rm o}, \qquad \eta=\sin 2\alpha\,E_{\rm o}. \tag{17.26}$$

In terms of the original Ernst potentials, the transformation (17.24) or (17.26) can also be stated in the equivalent form that, if $Z_{\rm o}$ is a solution of the vacuum Ernst equation (11.8), then a new electromagnetic solution satisfying (16.21) and (16.20) is given by

$$\begin{aligned} Z&=\frac{(1-a)+(1+a)Z_{\rm o}}{(1+a)+(1-a)Z_{\rm o}}=\frac{\sin^2\alpha+\cos^2\alpha\,Z_{\rm o}}{\cos^2\alpha+\sin^2\alpha\,Z_{\rm o}}\\ H&=\frac{\sqrt{1-a^2}\,(Z_{\rm o}-1)}{(1+a)+(1-a)Z_{\rm o}}=\frac{\sin\alpha\cos\alpha\,(Z_{\rm o}-1)}{\cos^2\alpha+\sin^2\alpha\,Z_{\rm o}}. \end{aligned} \tag{17.27}$$

It is of interest to note the similarity between the initial forms of these expressions and those of (17.9).

Proceeding in a similar fashion as in (17.10), it can be seen that the potentials for the new solution are given in terms of the vacuum potentials $Z_o = \Psi_o + i\Phi_o$ by

$$\Psi = \frac{4}{\Omega^2}\Psi_o \qquad \Phi = \frac{4a}{\Omega^2}\Phi_o \tag{17.28}$$

where, as in (17.9),

$$\begin{aligned} \Omega^2 &= |(1+a) + (1-a)Z_o|^2 \\ &= 4|\cos^2\alpha + \sin^2\alpha\, Z_o|^2. \end{aligned} \tag{17.29}$$

The expressions (17.27) may be substituted directly into (16.28), which then reduces exactly to (11.26). It follows that, if the initial solution Z_o satisfies the boundary conditions for colliding plane gravitational waves, then the new solution (17.25) also satisfies the required boundary conditions for the same metric function U given by

$$U = U_o. \tag{17.30}$$

It can be seen that the vacuum solution can be recovered when $a = 1$ or $\alpha = 0$, and that the original technique of Section 16.2 which is given by (16.33) or (16.34) is obtained when $a = 0$ or $\alpha = \pi/4$.

The transformation (17.24) was first used by Halilsoy (1989*a*) to obtain a new class of solutions with the familiar form for the vacuum solution given by

$$E_o = pt + iqz \tag{17.31}$$

where, as in (13.4), $p^2+q^2 = 1$. The resulting solutions have been analysed in detail. In the limit as $a \to 0$, this solution must reduce to that of Chandrasekhar and Xanthopoulos (1986*b*) as described in Section 13.3, which is the Kerr solution in the interaction region.

It is also of interest to note that it is a very simple matter to generalize the transformation (17.24), or (17.26), that has been used in this section. It can easily be seen that, if E_o is a solution of the vacuum equation (11.14), then a family of solutions of the electrovac equations (16.25) and (16.26) is given by

$$E = aE_o + b, \qquad \eta = cE_o + d \tag{17.32}$$

where a, b, c and d are arbitrary complex constants that are only required to satisfy the two conditions

$$a\bar{b} + c\bar{d} = 0, \qquad a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} = 1. \tag{17.33}$$

It can immediately be seen that this generation technique is not only a generalization of (17.24) and hence also of (16.33), but is also a generalization of the transformation (17.8) that was described in Section 17.1 and applied in Section 17.2. The generation technique given by (17.32) may also be expressed in the equivalent form

$$Z = \frac{1 + b + aE_o}{1 - b - aE_o}, \qquad H = \frac{d + cE_o}{1 - b - aE_o}. \tag{17.34}$$

By substituting (17.32) into (16.30), it can be seen that the boundary conditions for colliding plane waves are automatically satisfied for the new solution if they are satisfied for the initial solution E_o. It follows that, using the transformation (17.32), a large class of solutions describing colliding combined electromagnetic and gravitational waves can be obtained from any colliding gravitational plane wave solution.

17.5 Other solutions

It has been pointed out by Neugebauer and Kramer (1969) that, for any space-time having a non-null Killing vector, there are a number of invariance transformations for Einstein–Maxwell fields on the assumption that the electromagnetic field shares the space-time symmetry. These transformations can be considered to arise from the existence of scalar potentials and the symmetries of the associated potential space. This is well reviewed by Kramer *et al.* (1980). It is appropriate at the beginning of this section simply to summarize a number of results which follow from this, and which apply specifically to the colliding plane wave problem in which there is an additional space-like Killing vector.

In terms of the Ernst potentials introduced above, the invariance transformations may be stated in the form that if (Z_o, H_o) is a solution of the Ernst equations (16.22), then other solutions (Z, H) are given by

$$Z = \alpha\bar{\alpha} Z_o, \qquad H = \alpha H_o, \tag{17.35a}$$

$$Z = Z_o + ib, \qquad H = H_o, \tag{17.35b}$$

$$Z = \frac{Z_o}{1 + icZ_o}, \qquad H = \frac{H_o}{1 + icZ_o}, \tag{17.35c}$$

$$Z = Z_o + 2\bar{\beta} H_o + \beta\bar{\beta}, \qquad H = H_o + \beta, \tag{17.35d}$$

$$Z = \frac{Z_o}{1 + 2\bar{\gamma} H_o + \gamma\bar{\gamma} Z_o}, \qquad H = \frac{H_o + \gamma Z_o}{1 + 2\bar{\gamma} H_o + \gamma\bar{\gamma} Z_o} \tag{17.35e}$$

where α, β, and γ are complex constants and b and c are real constants. Together, these constants form the eight real parameters of the isometry group of the potential space. Clearly, these transformations can be used in any combination to generate new solutions of the Ernst equation.

It may immediately be observed that, in the absence of the electromagnetic field, (17.35*b*) is the transformation (12.17) and (17.35*c*) is the Ehlers transformation (12.13). Also, if the initial solution is purely gravitational so that $H_\mathrm{o} = 0$, then any combination of these transformations which includes (17.35*e*) will generate electromagnetic solutions from the initial vacuum solutions. Applied in this sense with $H_\mathrm{o} = 0$, (17.35*e*) is referred to as a Harrison transformation (Harrison, 1968).

It can also be shown that, if the initial solution $(Z_\mathrm{o}, H_\mathrm{o})$ satisfies the boundary conditions (16.28) for colliding plane waves, then this condition is also satisfied by a new solution (Z, H) obtained by any of the invariance transformations (17.35) for the same values of the parameters k_i. Thus, for any initial solution which is a colliding plane wave solution, further colliding plane wave solutions can be generated by these transformations. This result is very remarkable since, in the context of stationary axisymmetric space-times, these transformations do not always preserve asymptotic flatness and so are not of great practical importance in that situation.

It was initially pointed out by Kinnersley (1973)[2] that the group of symmetry transformations of the Einstein–Maxwell equations with a non-null Killing vector is the group $SU(2,1)$. This is the group of the transformations (17.35). It has been further described in terms of colliding plane wave solutions by Li and Ernst (1989).

It can now be seen that the approaches of Chandrasekhar and Xanthopoulos (1985*a*) described in Section 17.1 and of Halilsoy (1989*a*) as described in Section 17.4 are both equivalent to 'Ehlers–Harrison' transformations for particular choices of parameters.

Clearly a large class of further colliding plane wave solutions can be obtained using the above transformations. A four-parameter generalization of the Nutku–Halil solution has already been obtained by García-Díaz (1988) using this method. García-Díaz (1989) has also used a Harrison transformation to obtain a four-parameter generalization of the Ferrari–Ibañez solution. Further solutions have been generated in this way by Li and Ernst (1989) generalizing the solutions of Ernst, García-Díaz and Hauser (1987*a,b*, 1988), Ferrari, Ibañez and Bruni (1987*a,b*) and the Nutku–Halil solutuion.

There is also a large number of more general solution-generating techniques that can be applied to this situation. These are effectively generalizations of those mentioned briefly in Section 12.6, which were initially developed in the context of stationary axisymmetric space-times. For example, Cosgrove (1981) has found a Bäcklund transformation, expressed in terms of the Hauser–Ernst formalism, which applies to electrovac space-times with two Killing vectors.

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[2] See also Kramer *et al.* (1980) section 30.3.

It is finally appropriate here to refer to other non-diagonal solutions which describe colliding plane electromagnetic waves.

A five-parameter family of type D electrovac solutions has been given by Papacostas and Xanthopoulos (1989). In these solutions, the hypersurface $f+g=0$ in region IV is a quasi–regular singularity. The solution is in fact a special case of a family of solutions given by Debever (1971), Plebański (1975) and Plebański and Demiański (1976).[3] There is also a vacuum limit of this solution which describes a collision of gravitational waves.

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[3] See also Kramer *et al.* (1980) section 19.1.